Invariants and Covariants of the Cremona Cubic Surface

DISSERTATION

Submitted to the Board of University Studies of the Johns Hopkins University in Conformity with the Requirements for the Degree of Doctor of Philosophy

BY

C. P. SOUSLEY

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By C. P. Sousley.

Introduction.

If $a, b, \ldots, f$ are properly chosen cubic curves on P_6^2, *i. e.*, on six points in a plane, the cubic surface mapped on the plane by these curves may be given by the equation

$$a^3+b^3+\ldots.+f^3=0,$$

where the variables are subject to the relations

$$a+b+\ldots.+f=0,$$
$$\bar{a}a+\bar{b}b+\ldots.+\bar{f}f=0.$$

This form is known as the hexahedral cubic surface of Cremona.*

As to the invariants and covariants of the Sylvester pentahedral form of the surface much is known and given in explicit form in Salmon's geometry of three dimensions. Nothing, however, is known as to the invariants and covariants of this Cremona form.

The purpose of this paper is to obtain some of these invariants and covariants and to outline the further steps necessary for the determination of the invariants and linear covariants.

The results are important in determining the lines on a general cubic surface. In particular for the Cremona form, the equations of the lines are known. If, then, one can find a typical representation for the general cubic surface in terms of the Cremona form and can determine the irrational invariants for the Cremona form, then the required lines of the general surface can be determined from the known lines of the Cremona cubic surface.

This requires the calculation of the invariants and linear covariants for the Cremona form and their identification with the corresponding invariants and covariants of the general cubic surface.

The results obtained are valuable also for any study that involves the behavior of the lines of a cubic surface with reference to the covariants of the surface.

Section 1.

Given in S_3 the cubic surface $(\alpha x)^3 \equiv (\beta x)^3 \equiv \ldots.$ a known comitant $|\alpha\beta\gamma\xi|\ldots.(\delta x)\ldots.$ becomes $|\alpha\beta\gamma\eta\zeta\xi|\ldots.(\delta x)$, when the surface is taken in S_5, and the variables are subject to the conditions $\sum_1^6 \eta_1 x_1=0$, $\sum_1^6 \zeta_1 x_1=0$.

* *Mathematische Annalen*, Vol. XIII.

This process is known as the transference principle of Clebsch. The straightforward method for obtaining the corresponding comitant in S_3 would be to eliminate two variables, thus getting the surface in S_3, and for this form to calculate the comitant, but this would give an unsymmetrical result; whereas, by the Clebsch transference principle we obtain a symmetrical result.

The process of contravariant differentiation is not affected by going up to S_5 from S_3 in this way. This fact is noted immediately on performing corresponding operations in both dimensions.

Let us examine the degree of the invariants and covariants of the Cremona form. A covariant of a cubic surface of degree d, weight w, class c, order σ, has $3d$ symbols α; distributed in $w-c$ symbols $|\alpha\beta\gamma\delta|$, c symbols $|\alpha\beta\gamma\xi|$ and σ symbols (αx), whence $3d=4(w-c)+3c+\sigma$, or $3d=4w-c+\sigma$. These symbols become by Clebsch's principle of transference $|\alpha\beta\gamma\delta\eta\zeta|$, $|\alpha\beta\gamma\xi\eta\zeta|$, (αx), where for the Cremona cubic surface the coefficients α are now numerical and the coefficients η all are 1 and the coefficients ζ are $\overline{a}, \overline{b}, \ldots, \overline{f}$. Hence, if d' is the degree in $\overline{a}, \overline{b}, \ldots, \overline{f}$ of the covariant for the Cremona form, we have

$$d'=w=\tfrac{1}{4}(3d-\sigma+c). \qquad (1)$$

Hence for this canonical form, the invariants have the degree $d'=6, 12, 18, 24, 30, 75$; the linear covariants have the degree $d'=8, 14, 20, 32$.

The invariants of the surface being invariants of P_6^2, must be expressible in terms of the rational and symmetric functions of the irrational system $\overline{a}, \ldots, \overline{f}$ of P_6^2. The linear covariants of the surface can be expressed in terms of the functions

$$K_i=\overline{a}^i a+\overline{b}^i b+\ldots+\overline{f}^i f, \qquad (i=2, \ldots, 5). \qquad (2)$$

Let us now calculate for P_6^2, referred to a special coordinate system, the irrational system $\overline{a}, \ldots, \overline{f}$, and from these form the rational system $a_2, \ldots, a_6$,* which are the elementary symmetric functions of $\overline{a}, \ldots, \overline{f}$.

Section 2.

If we take P_6^2 in the prepared form

$$\left.\begin{array}{llllll} (1) & 1\ 0\ 0, & & (4) & 1\ 1\ 1, \\ (2) & 0\ 1\ 0, & & (5) & x\ y\ u, \\ (3) & 0\ 0\ 1, & & (6) & z\ t\ u, \end{array}\right\} \qquad (3)$$

we have (*loc. cit.*, p. 170)

$$\left.\begin{array}{ll} 3\overline{a}=\rho-3(ux+ut), & 3\overline{d}=\rho-3(uy+uz), \\ 3\overline{b}=\rho-3(ux+yz), & 3\overline{e}=\rho-3(uy+xt), \\ 3\overline{c}=\rho-3(ut+yz), & 3\overline{f}=\rho-3(uz+xt), \\ \multicolumn{2}{c}{\rho=u(x+y+z+t)+xt+yz.} \end{array}\right\} \qquad (4)$$

* Coble, "Point Sets and Allied Cremona Groups," *Trans. American Math. Soc.*, Vol. XVI (1915), p. 155; in particular § 5.

§ 3. General Solution When Motion Is in One Plane.

Let θ be the angle between OP and OQ so that $v\cos\theta = l\xi' + m\eta' + n\zeta'$. It will now be seen that if the associated directions OP, OP' are distinct and equally inclined to OQ, the motion of the electron is in one plane and the solution can be found. OP and OP' are equally inclined to OQ if $\cos\theta = \frac{v}{c}$ for then PP' is perpendicular to OQ. Also since $\lambda = c - v\cos\theta$ we have $\mu = c\lambda$, and so $-c\lambda' = 2(\overline{v}\,\overline{v}')$. On substitution in the equation $-\mu\lambda' = \lambda\{cp + (\overline{v}\,\overline{v}')\}$ of § 2 we get $-c\lambda' = cp + (\overline{v}\,\overline{v}')$ or $cp = (\overline{v}\,\overline{v}')$. Substituting in the first of the equations (A) this gives $\mu(\xi'' - cl') = -(\xi' - cl)(\overline{v}\,\overline{v}')$ which is at once integrable, giving $\xi' - cl = a(c^2 - v^2)^{\frac{1}{2}}$ where a is a constant of integration. Similarly, we obtain the equations $\eta' - cm = b(c^2 - v^2)^{\frac{1}{2}}$, $\zeta' - cn = d(c^2 - v^2)^{\frac{1}{2}}$ where b and d are constants of integration. On multiplication by ξ', η', ζ', respectively, and addition, these three integrals give $a\xi' + b\eta' + d\zeta' = 0$ (remembering $v^2 = c^2 - c\lambda$).

Similarly, on using ξ'', η'', ζ'' as multipliers, we get, since $(\overline{v}\,\overline{v}') = cp$,

$$a\xi'' + b\eta'' + d\zeta'' = 0.$$

Thus, the velocity and the acceleration of the electron are in the same plane; hence, if a solution of the fundamental equations can be found which satisfies the equation $c\cos\theta = v$, the motion of the electron is in one plane. Conversely, if the motion is plane it is easy to pick out a solution for which $c\cos\theta = v$. The theorem of associated directions, then, gives a second solution which also satisfies $c\cos\theta = v$. Knowing these two solutions the integration of the Riccatian equation of § 1 is effected by means of a single quadrature.

Take the plane of motion to be $z = 0$ so that $\zeta' = 0$, $\zeta'' = 0$. It is evident that $cl = \xi'$, $cm = \eta'$ satisfy the first two equations. These values of l, m satisfy $c\cos\theta = v$. To determine n we have the condition $l^2 + m^2 + n^2 = 1$, giving $cn = \pm(c^2 - v^2)^{\frac{1}{2}}$. Either of these values of n satisfies the third equation which is, in this case, $\mu n' = -cn(l\xi'' + m\eta'')$. Hence, the solutions which give $c\cos\theta = v$ are $cl = \xi'$, $cm = \eta'$, $cn = \pm\mu^{\frac{1}{2}}$.

To obtain the general direction of projection it is necessary to solve the Riccatian equation. In the case of plane motion this is somewhat simplified and is $2(c^2 - v^2)\frac{d\sigma}{d\tau} = c\phi'' - 2i\sigma(\xi'\eta'' - \xi''\eta') - c\sigma^2\psi''$ where $\sigma(1+n) = l + im$. Substituting for l, m, n the values just given we have the two particular solutions σ_1, σ_2 given by $c(1+\omega)\sigma_1 = \phi'$, $c(1-\omega)\sigma_2 = \phi'$ where $c\omega = (c^2 - v^2)^{\frac{1}{2}}$. It is easy to verify that σ_1, σ_2 satisfy the Riccatian equation.

The general solution of the Riccatian equation is, therefore,*

$$\sigma=\sigma_2+\frac{\sigma_1-\sigma_2}{1+Ae^{\int(\sigma_1-\sigma_2)Rd\tau}},$$

where A is a constant of integration and $2(c^2-v^2)R=-c\psi''$. A is, in general, complex since σ is complex.

Thus, when the motion is plane, the problem is reduced to the integration

$$\int(\sigma_1-\sigma_2)Rd\tau \text{ or } c\int\frac{(\xi''-i\eta'')\,d\tau}{(\xi'-i\eta')\sqrt{c^2-\xi'^2-\eta'^2}}.$$

The integration can always be effected if the scalar quantity v is constant, and in this case the integral is $[c/(c^2-v^2)^{\frac{1}{2}}]\log\psi'$, so that

$$\sigma=\sigma_2+\frac{\sigma_1-\sigma_2}{1+A\psi'^{\frac{1}{\omega}}}.$$

If $\eta=0$, $\eta'=0$, $\eta''=0$, it is evident that the integration can be carried out. This is the case of rectilinear motion which will be treated in detail. The remaining case of most interest is that of uniform motion in a circle and the solution is in this case given in terms of the trigonometric functions. In general, the explicit determination of the equations of the lines of force depends on the evaluation of the integral given above.

§ 4. PARTICULAR CASES OF PLANE MOTION.

(a) *Rectilinear Motion.*

Taking the x-axis as the line of motion we see from § 3 that the solution depends on the integration of $c\int\frac{\xi''d\tau}{\xi'\sqrt{c^2-\xi'^2}}$. In this particular case, however, the general solution can be obtained at once directly from the Riccatian equation which becomes $2(c^2-\xi'^2)\sigma'=c\xi''(1-\sigma^2)$.

This gives $\log\frac{1+\sigma}{1-\sigma}=\frac{1}{2}\log\frac{c+v}{c-v}+\log A$ (for $\xi'=v$). Write $A=\alpha+i\beta$ and we have $\frac{1+\sigma}{1-\sigma}=(\alpha+i\beta)\left(\frac{c+v}{c-v}\right)^{\frac{1}{2}}$. Whence,

$$\sigma=\{(\alpha+i\beta)(c^2-v^2)^{\frac{1}{2}}-(c-v)\}\div\{(\alpha+i\beta)(c^2-v^2)^{\frac{1}{2}}+(c-v)\}.$$

From this point we must distinguish between the cases $v\lesseqgtr c$.

Linear Motion When $v<c$.

$(c^2-v^2)^{\frac{1}{2}}$ is real and so if s denotes the conjugate of σ we have

$$s\{(\alpha-i\beta)(c^2-v^2)^{\frac{1}{2}}+(c-v)\}=(\alpha-i\beta)(c^2-v^2)^{\frac{1}{2}}-(c-v).$$

---

* Cohen, "Differential Equations," p. 175.

Forming the product σs we find

$$\sigma s\{(\alpha^2+\beta^2)(c^2-v^2)+(c-v)^2+2(c-v)(c^2-v^2)^{\frac{1}{2}}\alpha\}$$
$$=(\alpha^2+\beta^2)(c^2-v^2)+(c-v)^2-2(c-v)(c^2-v^2)^{\frac{1}{2}}\alpha,$$

and since $(1+\sigma s)l=\sigma+s$, $(1+\sigma s)im=\sigma-s$, $(1+\sigma s)n=1-\sigma s$ we have

$$Rl=(c+v)(\alpha^2+\beta^2)-(c-v),\ Rm=2\beta(c^2-v^2)^{\frac{1}{2}},\ Rn=2\alpha(c^2-v^2)^{\frac{1}{2}},$$

where $R=(c+v)(\alpha^2+\beta^2)+(c-v)$.

The lines of force at time t are, with these values for l, m, n,

$$x=\xi+c(t-\tau)l,\ y=c(t-\tau)m,\ z=c(t-\tau)n,$$

and α, β are two arbitrary real constants.

It is seen that $\alpha y=\beta z$ or the lines of force lie in planes through the direction of motion a result evident, à priori, from symmetry. If v is constant, so also are l, m, n, and we get the well-known result* that in the case of uniform motion in a right line the line of electric force at time t through any point x, y, z is got by joining x, y, z to the position of the electron at that time t. Since the lines of force are symmetrical round the direction of motion it is sufficient to consider the lines in the plane $z=0$, so that $\alpha=0$. For instance, in the case of simple harmonic motion we may write $\xi=a\cos p\tau$ and the lines of electric force in the xy-plane are $x=a\cos p\tau+c(t-\tau)l$, $y=c(t-\tau)m$, where

$$l\{\beta^2(c-ap\sin p\tau)+(c+ap\sin p\tau)\}=\beta^2(c-ap\sin p\tau)-(c+ap\sin p\tau),$$
$$m\{\beta^2(c-ap\sin p\tau)+(c+ap\sin p\tau)\}=2\beta(c^2-a^2p^2\sin^2 p\tau)^{\frac{1}{2}}.$$

The single arbitrary constant β gives the single infinity of lines in the plane. To get an idea of the shape of the lines take the particular line $\beta=1$. Then, $cl=-ap\sin p\tau$, $cm=(c^2-a^2p^2\sin^2 p\tau)^{\frac{1}{2}}$; put $t=0$ and $\tau=-\tau'$ so that τ' takes positive values. Then, $x=a\cos p\tau'+c\tau' l$, $y=c\tau' m$, where $cl=ap\sin p\tau'$. Thus, $x=a(\cos p\tau'+p\tau'\sin p\tau')$, $y=\tau'(c^2-a^2p^2\sin^2 p\tau')^{\frac{1}{2}}$, where τ' takes positive values. If a is small we have at distances far from the origin

$$x^2+y^2=c^2\tau'^2,\quad x=a(\cos p\tau'+p\tau'\sin p\tau').$$

The line of force is of an oscillatory character with increasing amplitude about the line $x=0$.

Case of Uniformly Accelerated Motion in a Straight Line.

Let g be the uniform acceleration and suppose the electron to start from rest at the origin at time $\tau=0$. Then, $\xi=\frac{1}{2}g\tau^2$, $v=g\tau$ so that

$Rl=\beta^2(c+g\tau)-(c-g\tau)$, $Rm=2\beta(c^2-g^2\tau^2)^{\frac{1}{2}}$, where $R=\beta^2(c+g\tau)+(c-g\tau)$. Consider the particular line $\beta=1$. Then, $R=2c$ and $cl=g\tau$ and the equations to the line of force are $x=\frac{1}{2}g\tau^2+(t-\tau)g\tau$, $y=(t-\tau)(c^2-g^2\tau^2)^{\frac{1}{2}}$.

* J. J. Thomson, *Phil. Mag.*, Vol. XI (1881), p. 229; O. Heaviside, *Phil. Mag.*, Vol. XXVII (1889), p. 324.

These equations take a somewhat simpler form if we transfer the origin to the position of the electron at time t. Then, $x=-\frac{1}{2}g(t-\tau)^2$ and taking $t-\tau$ as the parameter s we have $x=-\frac{1}{2}gs^2$, $y^2=s^2\{c-g(t-s)\}\{c+g(t-s)\}$, where s takes values from t to zero. On eliminating s this is a quartic curve

$$\left\{y^2+4x^2+\frac{2x}{g}(c^2-g^2t^2)\right\}^2+8gt^2x^3=0.$$

It is easy to see how the lines of force due to a point charge in rectilinear motion are altered in shape when the moving charge is accelerated or retarded. For example, imagine the charge to have been moving with uniform velocity for a long time and, then, to be suddenly affected with a retardation g, and consider, as above, the line $\beta=1$ for which $l=\frac{v}{c}$; the electron, being at $\tau=0$ at $x=0$, is reduced to rest at time $\tau=\frac{v}{g}$. Consider the line of force at this time. The part given by the aggregate of particles imagined projected at times $\tau<0$ is a straight line which would pass if produced through the point $x=\frac{v^2}{g}$. At the point where this straight line meets the line $cl=v$ through the origin, the line of force bends and passes through the point $x=\frac{v^2}{2g}$ in a direction perpendicular to the line of motion. At any later time t the line of force is made up of three parts: (1) A straight line part which goes, if produced, through $x=vt$, (2) a bent part beginning where this line meets the sphere $x^2+y^2+z^2=c^2t^2$ and ending at $x=\frac{v^2}{2g}$, $y=c\left(t-\frac{v}{g}\right)$, and (3) the part of the line $x=\frac{v^2}{2g}$ from this point to $y=0$. A similar result holds for the lines got from other values of β. It is evident that if the electron is stopped instantaneously we can get the lines of force in the following way: Draw the pencil of lines through $x=vt$, $y=0$. The sphere with centre at origin and radius ct cuts these lines in points where the lines of force bend suddenly, the remaining portions being the radii of this sphere. This explains how the discontinuity due to the sudden stoppage of an electron spreads out in a spherical wave. The commonly accepted theory of Röntgen rays is that they are pulses of this type due to the sudden stoppage of cathode particles. It is important to notice that in all cases the direction of projection is determined by the velocity at the point of projection and is independent of its acceleration.

If we give particular values to v, g, c, it is easy to draw the bent portion of the line of force due to the retarded part of the motion. For instance, take $v=4$, $g=c=8$; then the electron is reduced to rest in time $\tau=\frac{1}{2}$, its position at that time being $x=1$. Taking $\beta=1$ we have $l=\frac{v}{c}$. First take $t=\frac{1}{2}$, so that we are drawing the line for the instant when the electron is reduced to rest. Then, if $0<\tau<\frac{1}{2}$, $l=\frac{4-8\tau}{8}=\frac{1}{2}-\tau$, $\xi=4(\tau-\tau^2)$. The distance from ξ to the corresponding point on the line of force is $c(t-\tau)=4(1-2\tau)=8l$. It is found that the bent portion of the line of force resembles closely the part in the positive quadrant of an ellipse whose axes are in the ratio of 10 : 34. Having drawn the line of force at time $t=\frac{1}{2}$ we get the form of the same line at any subsequent time by producing the radii vectores from ξ to the corresponding point on the line of force at time $t=\frac{1}{2}$, a constant distance. This method of deducing the successive positions of a given line of force from the position at any one time follows from the equations of the lines of force and is applicable in the general case.

Motion in a Straight Line When $v>c$.

In the equation

$$\{(\alpha+i\beta)(c^2-v^2)^{\frac{1}{2}}+(c-v)\}\sigma=\{(\alpha+i\beta)(c^2-v^2)^{\frac{1}{2}}-(c-v)\}$$

$(c^2-v^2)^{\frac{1}{2}}$ is now a pure imaginary. Writing it in form $i(v^2-c^2)^{\frac{1}{2}}$ we see that s, the conjugate of σ, is given by the equation

$$\{(\alpha-i\beta)(v^2-c^2)^{\frac{1}{2}}+i(c-v)\}s=(\alpha-i\beta)(v^2-c^2)^{\frac{1}{2}}-i(c-v).$$

From these we obtain as before

$$Rl=(\alpha^2+\beta^2)(v+c)-(v-c),\quad Rm=2\alpha(v^2-c^2)^{\frac{1}{2}},\quad Rn=2\beta(v^2-c^2),$$

where $R=(\alpha^2+\beta^2)(v+c)+(v-c)$.

The lines of force lie in planes through the axis of x; putting $\beta=0$ we obtain the lines in the plane xy. It will be noticed that, for these, if $\alpha=1$, $l=\frac{c}{v}$, and so $\lambda=0$. Hence, in the case of rectilinear motion with constant velocity $v>c$ the generators of the cone $l=\frac{c}{v}$ are directions of projection. The cone with vertex at $x=\xi(t)$ and semi-vertical angle $\alpha=\sin^{-1}\frac{c}{v}$ accordingly

separates the regions where there are lines of electric force from those regions, where there are no lines of force. The generators of the cone are lines of electric force. This is a known result.* If v is not constant, these successive cones envelope a surface of revolution, the meridians on which are lines of electric force, and which enclose all the lines of force.

Motion in a Circle with Uniform Velocity.

Let p be the constant angular velocity and d the radius of the circle. Then, the plane of the circle being $z=0$, and the centre of the circle being the origin of coordinates, we have $\xi=a\cos p\tau$, $\eta=a\sin p\tau$, $\zeta=0$; whence, $\phi=ae^{ip\tau}$ and the particular solutions of the Riccatian equation are σ_1, σ_2, where

$$c(1+\omega)\sigma_1=iape^{ip\tau},\ c(1-\omega)=iape^{ip\tau} \text{ and } c^2\omega^2=c^2-v^2=c^2-a^2p^2.$$

On substituting these values in the general solution for plane motion it is found that

$$\sigma=ie^{ip\tau}\left[\frac{c(1+\omega)}{ap}+\frac{1}{(\alpha+i\beta)e^{\frac{-ip\tau}{\omega}}-ap/2c\omega}\right]\dagger.$$

If $v<c$, ω is real; if $v>c$, ω is a pure imaginary, so there will be two distinct solutions according as $v\gtrless c$.

(a) *Case When $v<c$.*

Since ω is real, s is got by merely changing the sign of i wherever it appears in the expression for σ. Adding we have

$$\sigma+s=-\frac{2c(1+\omega)}{ap}\sin p\tau-2\frac{P\sin p\tau+Q\cos p\tau}{R},$$

where P, Q, R are defined by the equations

$$P=\alpha\cos p\tau/\omega+\beta\sin p\tau/\omega-ap/2c\omega,$$
$$Q=\alpha\sin p\tau/\omega-\beta\cos p\tau/\omega,$$
$$R=(\alpha^2+\beta^2)+a^2p^2/4c^2\omega^2-ap/c\omega\,(\alpha\cos p\tau/\omega+\beta\sin p\tau/\omega).$$

Similarly, on subtracting we find

$$-i(\sigma-s)=2c(1+\omega)/ap\cos p\tau+2\frac{P\cos p\tau-Q\sin p\tau}{R},$$

and on multiplication

$$\sigma s=\frac{c^2(1+\omega)^2}{a^2p^2}+\frac{\frac{2c(1+\omega)}{ap}P+1}{R};$$

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* O. Heaviside, "Electrical Papers."

† The integral $\int\frac{\psi''}{\psi'\omega}d\tau$ of § 3 is, in this case, $\frac{1}{\omega}\log\psi'$ and $\psi=ae^{-ip\tau}$.

from these equations we obtain $R(1+\sigma s)=S$, $R(1-\sigma s)=T$, where S and T are defined by the equations

$$S=(\alpha^2+\beta^2)\frac{2}{1-\omega}+\frac{1-\omega}{2\omega^2}-\frac{2ap}{c\omega}\left(\alpha\cos\frac{p\tau}{\omega}+\beta\sin\frac{p\tau}{\omega}\right),$$

$$T=(\alpha^2+\beta^2)\frac{2\omega}{\omega-1}+\frac{1-\omega}{2\omega},$$

and hence l, m, n are given by

$$l\equiv\frac{\sigma+s}{1+\sigma s}=-\frac{ap}{c}\sin p\tau+\frac{2}{S}\left\{\left(P+\frac{ap}{2c\omega}\right)\omega\sin p\tau-Q\cos p\tau\right\},$$

$$m\equiv-i\frac{\sigma-s}{1+\sigma s}=\frac{ap}{c}\cos p\tau-\frac{2}{S}\left\{\left(P+\frac{ap}{2c\omega}\right)\omega\cos p\tau+Q\sin p\tau\right\},$$

$$n=\frac{1-\sigma s}{1+\sigma s}=\frac{T}{S},$$

and with these values of l, m, n the parametric equations to the lines of force are $x=a\cos p\tau+c(t-\tau)l$, $y=a\sin p\tau+c(t-\tau)m$, $z=c(t-\tau)n$.

By a suitable choice of the constants α, β so as to get particular lines of force these expressions may be considerably simplified. Thus, if we put $\alpha=0$, $\beta=0$, we have $l=-(ap/c)\sin p\tau$, $m=(ap/c)\cos p\tau$, $n=\omega$, and the line of force is given by the equations

$$x=a\cos p\tau-ap(t-\tau)\sin p\tau,\ y=a\sin p\tau+ap(t-\tau)\cos p\tau,\ z=c\omega(t-\tau).$$

It is obvious that it lies on the hyperboloid of revolution

$$(x^2+y^2)/a^2-p^2z^2/c^2\omega^2=1$$

which contains the circle of motion. Writing this equation in the form

$$(x/a-pz/c\omega)(x/a+pz/c\omega)=(1+y/a)(1-y/a),$$

we see that the directions of projection are generators of the hyperboloid. The form of the line of force is evident if we consider its projection on the plane of motion. The equations of the projection are

$$x=a\cos p\tau-ap(t-\tau)\sin p\tau,\ y=a\sin p\tau+ap(t-\tau)\cos p\tau,\ z=0;$$

giving

$$(x-a\cos p\tau)+(y-a\sin p\tau)\tan p\tau=0$$

$$\text{and } (x-a\cos p\tau)^2+(y-a\sin p\tau)^2=a^2p^2(t-\tau)^2.$$

These are the equations to an involute of the circle $x^2+y^2=a^2$; the involute meeting the circle at the position of the electron at time $\tau=t$. To obtain, then, a line of electric force due to an electron moving with velocity $v<c$ in a circle we unwrap a string which is wound round the circle, the tracing point at the end of the string leaving the circle at the position of the electron at time t. Having obtained this involute we project it by lines parallel to the axis of z on the hyperboloid $(x^2+y^2)/a^2-p^2z^2/c^2\omega^2=1$.

Lines of Force in the Plane of Motion.

To obtain these put $n=0$. Then, $T=0$ or $(\alpha^2+\beta^2)=(\overline{1-\omega}/2\omega)^2$, and so we may write $2\omega\alpha=(1-\omega)\cos\theta$, $2\omega\beta=(1-\omega)\sin\theta$ and these give $\omega^2 S=(1-\omega)k$ where $k=1-(ap/c)\cos(p\tau/\omega-\theta)$, and, finally,

$$kl=\sin p\tau\cos(p\tau/\omega-\theta)-(ap/c)\sin p\tau-\omega\cos p\tau\sin(p\tau/\omega-\theta),$$
$$km=-\cos p\tau\cos(p\tau/\omega-\theta)+(ap/c)\cos p\tau-\omega\sin p\tau\sin(p\tau/\omega-\theta).$$

The single constant θ gives the single infinity of lines in the plane of motion; from symmetry we need only consider a particular value of θ, say $\theta=0$; then,

$$kl=\sin p\tau\cos p\tau/\omega-(ap/c)\sin p\tau-\omega\cos p\tau\sin p\tau/\omega,$$
$$km=-\cos p\tau\cos p\tau/\omega+(ap/c)\cos p\tau-\omega\sin p\tau\sin p\tau/\omega,$$

where the Doppler factor k is now $1-(ap/c)\cos p\tau/\omega$.

Circular Motion When $v>c$.

ω is now a pure imaginary and so it may be written $=i\theta$ where θ is real. The functions σ and s are given by the equations

$$\sigma=ie^{ip\tau}\left\{\frac{c(1+i\theta)}{ap}+\frac{1}{(\alpha+i\beta)e^{-p\tau/\theta}+iap/2c\theta}\right\},$$
$$s=-ie^{-ip\tau}\left\{\frac{c(1-i\theta)}{ap}+\frac{1}{(\alpha-i\beta)e^{-p\tau/\theta}-iap/2c\theta}\right\},$$

whence

$$D(\sigma+s)=-D\left(\frac{2c}{ap}\sin p\tau+\frac{2c\theta}{ap}\cos p\tau\right)+2e^{-p\tau/\theta}\{\beta\cos p\tau-\alpha\sin p\tau\}+\frac{ap}{c\theta}\cos p\tau,$$

where

$$D=\alpha^2e^{-2p\tau/\theta}+\left\{\beta e^{-p\tau/\theta}+\frac{ap}{2c\theta}\right\}^2.$$

Similarly,

$$-iD(\sigma-s)=\frac{2c}{ap}D(\cos p\tau-\theta\sin p\tau)+2e^{-p\tau/\theta}\{\alpha\cos p\tau+\beta\sin p\tau\}+\frac{ap}{c\theta}\sin p\tau$$

$$\text{and }\sigma s=1+\frac{2c}{apD}e^{-p\tau/\theta}(\alpha-\theta\beta).$$

Using these values we obtain, after some reduction, and on writing

$$F=(\alpha^2+\beta^2)e^{-2p\tau/\theta}-a^2p^2/4c^2\theta^2,$$
$$G=(\alpha^2+\beta^2)e^{-2p\tau/\theta}+a^2p^2/4c^2\theta^2+\frac{c}{ap}e^{-p\tau/\theta}(\alpha+\beta/\theta),$$

the equations

$$l=-\frac{c}{ap}\left[\sin p\tau+\frac{\theta\{F\cos p\tau+\frac{c}{ap}e^{-p\tau/\theta}(\theta\alpha+\beta)\sin p\tau\}}{G}\right],$$

$$m=\frac{c}{ap}\left[\cos p\tau-\frac{\theta\{F\sin p\tau-\frac{c}{ap}e^{-p\tau/\theta}(\theta\alpha+\beta)\cos p\tau\}}{G}\right],$$

$$n=\frac{c}{ap}e^{-p\tau/\theta}\frac{(\theta\beta-\alpha)}{G}.$$

If this collineation be written as

$$2^7 K = p_{11}\xi_1 + p_{22}\xi_2 + \ldots + p_{66}\xi_6, \tag{32}$$

we find

$$\left.\begin{aligned} p_{11} = {} & x_1[d_1 + b_1 d_2 + b_1^2 d_3 + b_1^3 d_4 + b_1^4 d_5 + b_1^5 d_6] \\ & + K_2[f_{2,0} + b_1 f_{2,1} + b_1^2 f_{2,2} + b_1^3 f_{2,3} + b_1^4 f_{2,4} + b_1^5 f_{2,5}] \\ & + K_3[f_{3,0} + b_1 f_{3,1} + b_1^2 f_{3,2} + b_1^3 f_{3,3} + b_1^4 f_{3,4} + b_1^5 f_{3,5}] \\ & + K_4[f_{4,0} + b_1 f_{4,1} + b_1^2 f_{4,2} + b_1^3 f_{4,3} + b_1^4 f_{4,4} + b_1^5 f_{4,5}] \\ & + K_5[f_{5,0} + b_1 f_{5,1} + b_1^2 f_{5,2} + b_1^3 f_{5,3} + b_1^4 f_{5,4} + b_1^5 f_{5,5}], \end{aligned}\right\} \tag{33}$$

where

$$\left.\begin{aligned} d_1 = {} & 2^4\cdot 3^2\cdot 53\, I_1^2 + 2^4\cdot 3^3\cdot 53\, a_3^2 I_1 - 5\cdot 10181\, a_2 q_4 I_1 + 2^4\cdot 3^4\cdot 5\, q_4^3 \\ & + 2^2\cdot 13\cdot 1609\, a_2^2 q_4^2 - 3\cdot 7\cdot 2911\, a_2 a_3^2 q_4 + 5^2 a_2^3 I_1 + 3\cdot 5^2 a_2^3 a_3^2 - 2^2\cdot 5^2 a_2^4 q_4, \\ d_2 = {} & 6(2^4\cdot 3\cdot 359\, q_5 I_1 - 2^4\cdot 3^2\cdot 53\, a_2 a_3 I_1 - 3^2\cdot 5^2\cdot 257\, a_2 q_4 q_5 + 2^7\cdot 3^2\cdot 5^2 a_3^2 q_5 \\ & + 5^2 a_2^3 q_5 + 2^4\cdot 3^4\cdot 5\, a_3 q_4^2 + 3\cdot 5\cdot 1439\, a_2^2 a_3 q_4 - 5^2 a_2^4 a_3), \\ d_3 = {} & 3(-2^4\cdot 3^2\cdot 53\, q_4 I_1 + 2^4\cdot 461\, a_2^2 I_1 + 2^9\cdot 3^2\cdot 5^2 q_5^2 - 2^4\cdot 3\cdot 2281\, a_2 a_3 q_5 \\ & + 14153\, a_2 q_4^2 + 2^4\cdot 3^5\cdot 5\, a_3^2 q_4 - 2\cdot 12097\, a_3^2 q_4 - 2^4\cdot 3^2\cdot 7^2 a_2^2 a_3^2 + 5\cdot 101\, a_2^5), \\ d_4 = {} & 12(-2^4\cdot 3^2\cdot 53\, a_3 I_1 - 2^4\cdot 3\cdot 7\cdot 37\, q_4 q_5 + 2^2\cdot 3\cdot 1049\, a_2^2 q_5 \\ & + 5\cdot 3473\, a_2 a_3 q_4), \\ d_5 = {} & 12(2^3\cdot 3^2\cdot 83\, a_2 I_1 - 2^4\cdot 3^2\cdot 359\, a_3 q_5 - 2^4\cdot 3\cdot 41\, q_4^2 - 3\cdot 5539\, a_2^2 q_4 + 25\, a_2^4), \\ d_6 = {} & 288(1621\, a_2 q_5 + 2^3\cdot 3^2\, a_3 q_4). \\ f_{2,0} = {} & 6(17\, q_4 I_1 - 563\, a_2^2 I_1 - 3^2\cdot 5\, q_5^2 - 2^3\cdot 3^2\cdot 5\, a_2 a_3 q_5 + 2^3\cdot 59\, a_2 q_4^2 \\ & - 3\cdot 487\, a_3^2 q_4 + 2^3\cdot 5\cdot 89\, a_2^3 q_4 - 2^2\cdot 3\cdot 143\, a_2^2 a_3^2 - 2^4\cdot 3\cdot 5\, a_2^5), \\ f_{3,0} = {} & 18(421\, a_3 I_1 + 2\cdot 3\cdot 23\, a_2^2 q_5 - 2\cdot 5\cdot 127\, a_2 a_3 q_4 + 2^3\cdot 3\cdot 53\, a_3^3), \\ f_{2,1} = {} & 6(-3\cdot 7^3 + 3\cdot 19\, q_4 q_5 - 3\cdot 7\cdot 167 + 2\cdot 11\cdot 251 - 2^3\cdot 3^2\cdot 53 \\ & + 2^3\cdot 3\cdot 143\, a_2^3 a_3), \\ f_{4,0} = {} & 12(-2^2\cdot 11\cdot 13\, a_2 I_1 + 2\cdot 3^2 a_3 q_5 + 2^3\cdot 3^2\cdot 5\, q_4^2 + 2^3\cdot 5\cdot 59\, a_2^2 q_4 \\ & - 2^2\cdot 3\cdot 143\, a_2 a_3^2), \\ f_{3,1} = {} & 12(2^2\cdot 157 + 2\cdot 3^2\cdot 5\cdot 41 - 2\cdot 3^2\cdot 5 - 2\cdot 1039 - 2^2\cdot 3\cdot 143), \\ f_{2,2} = {} & 12(619 - 3^2\cdot 829 - 2^4\cdot 3 - 2\cdot 667 + 2^2\cdot 3\cdot 461 - 2\cdot 3\cdot 143\, a_2^4), \\ f_{5,0} = {} & 36(3\cdot 5^2 a_2 q_5 + 3\cdot 7\cdot 11\, a_3 q_4), \\ f_{4,1} = {} & 36(-1093 + 3^2\cdot 47 + 2^3\cdot 143\, a_2^2 a_3), \\ f_{3,2} = {} & 36(7\cdot 389 - 3\cdot 11\cdot 17 - 2^2\cdot 3\cdot 53), \\ f_{2,3} = {} & 36(5^2\cdot 101 - 3^2\cdot 79 - 2^2\cdot 3\cdot 53), \\ f_{5,1} = {} & 12(2\cdot 3\cdot 197\, I - 2^3\cdot 7^2\cdot 11\, a_2 q_4 + 2^3\cdot 3^2\cdot 53\, a_3^2), \\ f_{4,2} = {} & 12(2\cdot 3\cdot 197 - 2^2\cdot 5\cdot 137 + 2^3\cdot 3^2\cdot 53 - 2^2\cdot 3\cdot 143\, a_2^3), \\ f_{3,3} = {} & 12(2^3\cdot 3\cdot 53 - 2^2\cdot 19\cdot 31 - 2^3\cdot 3^2\cdot 53 + 2^2\cdot 3\cdot 143), \\ f_{2,4} = {} & 12(2^4\cdot 3\cdot 5^2 - 2^2\cdot 5^2\cdot 31 + 2^3\cdot 3^2\cdot 53 - 2^2\cdot 3\cdot 143), \\ f_{5,2} = {} & 2^4\cdot 3^3\cdot 317\, q_5 - 2^6\cdot 3^2\cdot 151\, a_2 a_3, \quad f_{4,3} = 2^4\cdot 3^3\cdot 307\, q_5 - 2^9\cdot 3^2 a_2 a_3, \\ f_{3,4} = {} & 2^4\cdot 3^5\cdot 5\cdot 7\, q_5 - 2^9\cdot 3^2 a_2 a_3, \quad f_{2,5} = 2^3\cdot 3^3\cdot 619\, q_5 - 2^6\cdot 3^2\cdot 151\, a_2 a_3, \\ f_{5,3} = {} & 2^6\cdot 3^2\cdot 11\, q_4 + 2^5\cdot 3^2\cdot 143\, a_2^2, \quad f_{4,4} = -2^6\cdot 3^2\cdot 13\, q_4 - 2^5\cdot 3^2\cdot 143\, a_2^2, \\ f_{3,5} = {} & 2^5\cdot 3^2\cdot 7\, q_4 + 2^5\cdot 3^2\cdot 143\, a_2^2, \quad f_{5,4} = f_{4,5} = 2^6\cdot 3^3\cdot 53\, a_3, \\ f_{5,5} = {} & 2^6\cdot 3^2\cdot 143\, a_2. \end{aligned}\right\} \tag{34}$$

Thus the coefficient of ξ_1 in the collineation can be expressed in terms of x_1, K_2, K_3, K_4, K_5 linearly with coefficients which are functions of a_2, a_3, q_4, q_5, I_1 and $b_1, b_1^2, \ldots, b_1^5$.

As for the second invariant the sum $K_{1,1}+\ldots+K_{6,6}$ if not identically zero would furnish the self-apolarity invariant of the collineation which we could define to be I_2.

The result of operating with C on Q is a $C_{19,0,1}$, *i. e.*, the second linear covariant L_2. Hence,

$$5/2\, L_2 = \sum_{(1)}^{6} (6 C_1 S_1 + \sum_{(2)}^{5} C_2 S_{12})\, x_1 . \tag{35}$$

The collineation K sends L_1 into the third linear covariant L_3. This likewise is sent into the fourth linear covariant L_4 by the same collineation.

The three remaining invariants I_3, I_4, I_5 are obtained by operating with C on L_1, L_2, L_3, respectively.

VITA.

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